Extending Randomized Single Elimination Bracket to Multiple Prize Vectors

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Abstract

We extend Randomized Single Elimination Bracket (RSEB) [3] to tournaments with multiple prizes, creating two new tournament ranking rules: *Randomized Complete Bracket (RCB)* and *Randomized Recursive Bracket (RRC)*. We prove various guarantees for manipulability and fairness under these rules. We first show that Randomized Complete Bracket is 2-SNM-$\frac{1}{2}$ and that both RCB and RRB are 2-SNM-$\frac{1}{3}$ for a subclass of prize vectors we call *Binary Power-of-Two Prize Vectors*. We then show that both rules are manipulable under the Borda vector presented in [1] and that neither is cover-consistent or consistent under expectation. Finally, we provide a promising partial proof and empirical results towards showing that Randomized Complete Bracket is 2-SNM-$\frac{1}{3}$ for all prize vectors.

1 Introduction

A tournament rule maps the results of $\binom{n}{2}$ pairwise matches by n teams to a (possibly random) ranking on those teams. Tournaments have been studied previously in the context of algorithmic game theory, where previous work has examined how to design tournament rules that are both fair and minimally manipulable (see, for instance, [1, 2, 3]). We continue this work by building off of [1] which introduced the idea of tournaments with multiple winners who receive (potentially) different prizes (prior work had considered tournaments with a single winner). We extend the tournament rule Randomized Single Elimination Bracket (RSEB) described in [3] (which considers tournaments with a single winner) to two new tournament rules, Randomized Complete Bracket (RCB) and Randomized Recursive Bracket (RRB), which can award prizes to multiple teams by determining a ranking over those teams. We then examine RCB and RRB to show the following (see definitions for terms below in Section 1.1):

- RCB is 2-SNM-$\frac{1}{2}$, meaning any two teams cannot increase their expected collective prize winnings by more than $\frac{1}{2}$
- RCB and RRB are 2-SNM-$\frac{1}{3}$ (which is optimal over Condorcet-Consistent tournament rules) for the BPoT ("Binary Power-of-Two") subclass of prize vectors
- RCB and RRB are not consistent under expectation and are not cover-consistent
- The Borda prize vector is manipulable under RCB and RRB (unlike the tournament rule in [1])

1.1 Preliminaries

The following definitions align with current literature on tournament rules. Definitions 4, 8, and 9 are specific to this paper. All other definitions are from [1] with minor revisions. Note that for most of our results (all except the ones in Section 3: Fairness), we assume that the number of teams $n = 2^k$ for some positive integer k for ease of calculation, though it seems likely that our results would extend more generally to other tournaments.

Definition 1 (Tournament). *A tournament T on n teams is a complete, directed graph on n vertices whose edges denote the outcome of a match between two teams. Team i beats team j if the edge between them points from i to j.*

Definition 2 (Tournament Ranking Rule). *A tournament ranking rule r is a function that maps tournaments T to a distribution over rankings σ, where $\sigma_T^r(u)$ denotes the random variable that is the ranking of team u under rule r, applied to tournament T. We will refer to tournament ranking rules simply as tournament rules.*

Definition 3 (Prize Vector). *A prize vector is a vector $\vec{p} \in [0,1]^n$ such that $p_i \geq p_{i+1}$ for all i. Under a prize vector $\vec{p}$, the team ranked i^{th} receives prize p_i.*

Definition 4 (Binary Power-of-Two (BPoT) Prize Vectors). *A vector $\vec{p} \in \{0,1\}^n$ with $n = 2^m$ total entries is BPoT if for 2^k total entries, $p_i = 1$, for some integer $0 \leq k \leq m$.*

Definition 5 (Borda Prize Vector). *The Borda prize vector for a tournament with n teams is the prize vector with n entries where $p_i = \frac{n-i}{n-1}$.*

Definition 6 (Bracket). *A bracket B on $n = 2^h$ teams is a complete binary tree of height h whose leaves are labelled with some permutation of the n teams.*

Definition 7 (Randomized Single Elimination Bracket (RSEB)). *First we select a bracket B uniformly at random from all $n!$ possibilities. The outcome of a bracket B under an RSEB tournament T is the labelling of internal nodes of B where each node is labelled by the winner of its two children under T. The winner of B under T is the label of the root of B under this labelling. If n is not a power of 2, we define the random single elimination bracket rule on n teams by introducing $2^{\lceil log_2 n \rceil} - n$ dummy teams who lose to all of the existing n teams. Note RSEB does not produce a ranking of teams.*

Definition 8 (Randomized Recursive Bracket (RRB)). *For a tournament T on n teams (where n is power of 2), let M be matches corresponding to a uniformly random perfect matching (i.e. exactly $n/2$ matches are played, and every team plays in exactly one match). Let W be the winners and L be the losers. Let $T|_S$ be the induced subgraph of T on teams in set S. Recur on $T|_W$ with prize vector $\vec{p}_W = \langle p_1, ..., p_{|W|} \rangle$ and, separately, recur on $T|_L$ with prize vector $\vec{p}_L = \langle p_{|W|+1}, ..., p_n \rangle$. If n is not power of 2, then in the first round, let $n' := 2^{\lceil \log_2 n \rceil}$.*
Create $n' - n$ dummy teams who all lose to the original n teams, and match all of these dummy teams to actual teams in the first round.

Definition 9 (Randomized Complete Bracket (RCB)). *First select an ordering of teams $t_1, t_2, ..., t_n$ uniformly at random from all $n!$ possibilities. In the first round, have each team of odd parity t_i play the t_{i+1}th team in the ordering. At each round of the bracket, recur on the induced ordering of winners and ordering of losers. Note that this produces the following primary constraint of RCB: given two pairs a_1 beats b_1 and a_2 beats b_2 at the kth round of the bracket, if a_1 plays a_2 in the $k+1$th round, b_1 plays b_2 in the $k+1$th round.*
Note that in the determinstic setting, an RCB bracket and final ranking is well-defined by the initial ordering of teams since all randomization is done at the initialization of the bracket when producing the initial ordering.

Definition 10 (Nested Randomized King of the Hill (NRKotH)). *In each round, a random "pivot" team is chosen - teams that beat the pivot are placed into the winner set while teams that lose to the pivot are placed in the loser set. All teams in the loser set are ranked lower than those in the winner set. The next round then runs NRKotH on the winnder set and loser set - the result of this recursion is the ranking that the tournament rule outputs.*
NRKotH is the main tournament rule studied in [1] which is similar to Quicksort.

Definition 11 (Strongly Non-Manipulable (k-SNM-α)). *A tournament rule r is k-SNM-α if any set of k teams cannot increase their expected collective prize winnings by more than α.*

Definition 12 (Condorcet-Consistent). *Team i is a Condorcet winner of a tournament T if i beats every other team (under T). A tournament ranking rule r is Condorcet-consistent if for every tournament T with a Condorcet winner i, $P[\sigma_T^r(i) = 1] = 1$ (that team is ranked first with probability 1).*

Definition 13 (Consistent Under Expectation). *Let $w_T(u)$ be the set of teams that u defeats in tournament T. Let $\sigma_T^r(u)$ be the random variable that is the ranking of team u under rule r, applied to tournament T. A tournament rule is consistent under expectation if for all n, all tournaments T on n teams, and all u,*

$$\mathbb{E}[\sigma_T^r(u)] \quad = \quad n - |w_T(u)|$$

In other words, if team u beats k teams in T, u is expected to rank above k teams.

Definition 14 (Cover-Consistent). *Team i covers team j in tournament T if i beats j, and i beats every team that j beats. A tournament ranking rule is Cover-Consistent if for all T, and all i, j such that i covers j in T, $P[\sigma_T^r(i) < \sigma_T^r(j)] = 1$. That is, whenever i covers j in T, rule r applied to T should output a ranking where i is ahead of j with probability* 1.

1.2 Related Work

Several tournament rules have been shown to be 2-SNM-$\frac{1}{3}$ on the one-winner prize vector (that is, $\langle 1, 0, 0, ...\rangle$). The main result of a paper from 2016 draws our attention to the RSEB tournament rule as one such rule. [3]

Theorem 1 ([3] Theorem 3.3). *RSEB is 2-SNM-$\frac{1}{3}$ on the $\langle 1, 0, 0, ...\rangle$ prize vector.*

Notably, RSEB does not extend as previously defined to arbitrary prize vectors since it does not produce a ranking over all teams other than the winner. However, prior work [1] has shown that it is possible to produce tournament rules over arbitrary prize vectors that are 2-SNM-$\frac{1}{3}$ and that it is the best possible guarantee of pairs of colluding teams.

Theorem 2 ([1] Theorem 3.1). *NRKoTH is 2-SNM-$\frac{1}{3}$. That is, for any prize vector in $[0, 1]^n$, and any underlying tournament T, no two teams can manipulate their match to gain expected prize money more than $1/3$. Moreover, this is the best possible guarantee of any Condorcet-Consistent tournament ranking rule.*

We extend this analysis of tournament rules that produce complete rankings for arbitrary prize vectors by generalizing RSEB to create RCB and RRB. Another possible avenue for similar analysis would be a generalization of Randomized Death Match, which described in [2] and is also 2-SNM-$\frac{1}{3}$ on the $\langle 1, 0, 0, ...\rangle$ prize vector ([2] Theorem 5.2). We choose to focus on the RSEB generalization because of its desirable property that every team plays the same number of matches, a property not seen in NRKoTH but potentially necessary to practical tournaments. We leave analysis of Randomized Death Match to future work.

In addition to the desirable 2-SNM-$\frac{1}{3}$ property, NRKoTH is strongly non-manipulable on the Borda prize vector, which motivated our investigation into RCB and RRB on the Borda vector. Specifically, [1] Theorem 4.1 showed that NRKotH is k-SNM-0 for all $k \leq n$ on the Borda prize vector, which is a very strong property. Additionally, NRKoTH is consistent under expectation ([1] Theorem 4.3). We attempt to analyze the presence of these desirable properties on extensions of RSEB to arbitrary prize vectors.

2 Manipulability

2.1 Concrete Results

Lemma 3. *A pair of colluding teams can increase their collective prize winnings by at most* 1 *under RCB and RRB*

Proof. Consider teams i and j. Say j beats i in tournament T, and let T' be the tournament identical to T except j throws the match against i. Consider any instance of RCB or RRB run on T and T'. If i and j do not meet in this instance, their winnings are the same under T and under T'. Now, say i and j do meet. Let p_m be the worst prize the winner of their match could attain, and p_{m+1} be the best prize the loser of their match could attain. Then, the worst collective winnings i and j could have in T is $p_m + p_n$, and the best collective winnings i and j could have in T' is $p_1 + p_{m+1}$. Thus, the most they could increase their collective prize winnings by is $p_1 + p_{m+1} - p_m - p_n$, which is at most $p_1 - p_n$ (because $p_m \geq p_{m+1}$), which is at most 1. □

Lemma 4. *A pair of colluding teams cannot increase their collective prize winnings if they win against identical sets of teams (disregarding each other) under RCB or RRB.*

Proof. A similar lemma is proved as Lemma 3.3 in [1] for NRKotH, which we present here for RCB and RRB. Suppose that i and j are a pair of colluding teams, each of which wins against the teams in the set W and loses against the set of teams in L, where $W \cup L \supset [n] \setminus \{i, j\}$. Then we can see that the outcome of the (i, j) match has no effect on the collective prize winnings of i and j. If i and j do not play each other in a given play through of RRB or RCB, then the outcome of a match between them clearly does not affect their collective prize winnings. If i and j do play each other, let w denote the winner and ℓ denote the loser of that match. In subsequent rounds of RCB or RRB, note that the identity of w and the identity of ℓ will not affect whether w and ℓ win or lose matches since i and j defeat exactly the same set of teams. Thus, w will attain a certain prize p_w and ℓ will attain a certain prize p_ℓ, so i and j attain collective prize winnings of $p_w + p_\ell$ regardless of who wins the (i, j) match. □

Lemma 5. *In RCB, a pair of colluding teams i and j cannot gain from collusion if $s_i \in [1, n/2]$ and $s_j \in [n/2 + 1, n]$ (or vice versa) where s_i is the slot team i is assigned to in the initial bracket. In other words, i and j cannot gain from collusion if they are in different halves of the bracket.*

Proof. Let i and j be the teams attempting to collude. Note that if i and j only play each other in one of the last games of the bracket (i.e. each of their $\log_2 n$th games), i and j cannot gain from collusion since the sum of their prizes is the same no matter which team wins. In other words, i and j will always get total winnings $p_k + p_{k+1}$ for some k. Additionally, note that if i and j are not in the same half of the bracket (slots $[1, n/2]$ or $[n/2 + 1, n]$), i and j are guaranteed not to play each other until the last games (if ever). Thus, i and j cannot gain from collusion if they are in different halves of the bracket. □

Theorem 6. *RCB is 2-SNM-$\frac{1}{2}$.*

Proof. First, we find the number of possible initializations of RCB in which colluding teams i and j are not placed in the same half of the graph. For a fixed location of i in the first half of the graph, there are $n/2$ possible locations for j in the second half of the graph. Since there are $n/2$ slots on each half of the graph, and since i could be on either the first or second half of the graph, this means there are $2 * \left(\frac{n}{2}\right)^2$ possible positions for i and j such that they are on opposite halves of the bracket. There are then $(n-2)!$ permutations of the remaining teams to assign them to the remaining slots. Thus, there are overall $\frac{n^2}{2}(n-2)! = \frac{n}{2(n-1)}n!$ possible initializations such that i and j are on different halves of the bracket.

Let k be the number of initializations of the bracket that cannot be manipulated by teams i and j. By lemma 2, this implies there are at least $k \geq \frac{n}{2(n-1)}n! \geq \frac{n!}{2}$ initializations for RCB that cannot be manipulated by i and j. Additionally, note that the number of possible starting brackets is $n!$. Thus, the fraction of manipulable games (which is equal to the probability that i and j can collude and increase their collective winnings) is at most $\frac{n!/2}{n!}$. Thus, by lemma 1, the expected total gains from collusion are at most $1 * \frac{n!/2}{n!} = 1/2$. In summary, the total number of non-manipulable brackets k is $k \geq \frac{n}{2(n-1)}n! \geq \frac{n!}{2}$ and the total number of possible brackets is $n!$, so the fraction of possible brackets that are non-manipulable is $\frac{k}{n!} \geq \frac{n!}{2n!} = \frac{1}{2}$ and by lemma 1, in each of the remaining $n! - k \leq \frac{n!}{2}$ manipulable games, i and j can only increase their winnings by at most 1.

Thus, a pair of teams i and j cannot increase their expected total winnings by more than $1/2$. □

Theorem 7. *RCB and RRB are 2-SNM-$\frac{1}{3}$ on BPoT prize vectors.*

Proof. For this proof, we will slightly abuse notation by saying a team i "wins" if they get a prize of 1 and "loses" if they get a prize of 0. This is an intuitive restatement of our prize vector, since any binary prize vector can be alternatively written as a threshold rank at which teams become "winners" of prize 1.

It's here that it becomes important we are specifically considering binary power-of-two vectors. Consider the following lemma:

Lemma 8. *Consider an RCB or RRB tournament using a BPoT prize vector with 2^k entries of which 2^i have a prize of 1. A given team "wins" (gets a prize of 1) if and only if they win their first $k-i$ matches*

Proof. Because prize vectors are monotonically non-increasing, getting a prize of 1 under a tournament with a BPoT prize vector amounts to being in the top $\frac{2^i}{2^k} = 2^{-(k-i)}$ of teams. RCB and RRB subdivide teams into two groups (winners and losers) each round, such that the top 2^{-r} of teams are always the set of teams that won their first r matches, for $r \leq k$. Thus, the set of teams who get a prize of 1 under RCB and RRB with BPoT prize vectors are exactly those people that win their first $k-i$ matches □

By Lemma 11, note that if a team loses any of their first $k-i$ matches, they get a reward of 0 (i.e. they are "eliminated"); otherwise, they get a reward of 1. In this way, we can see that RRB or RCB on a BPoT prize vector with 2^i ones is equivalent to splitting the 2^k competitors into 2^i subgroups of size 2^{k-i} and running RSEB on each of these subgroups, where the winner of each subgroup receives a prize of 1. Since RSEB is 2-SNM-$\frac{1}{3}$, it follows that no two teams within a subgroup can improve their collective winnings by more than $\frac{1}{3}$ in expectation, so no two teams in the entire tournament can improve their collective winnings by more than $\frac{1}{3}$ in expectation, proving the theorem.

We note that there is an alternative proof for this theorem that simply modifies Theorem 3.3 from [3] slightly, but we omit it because it is much longer and more complex. □

As mentioned above, previous work [1] shows that a certain tournament rule, Nested Randomized King of the Hill, is nonmanipulable by any number of teams on the Borda prize vector p^* which has entries $p_i^* = \frac{n-i}{n-1}$ such that the team who comes in first place gets a reward of 1 and the team that comes in last gets a reward of 0. As such, it is reasonable to consider whether RCB might have a similar property - a counterexample can show that this is not the case.

Theorem 9. *RCB and RRB are manipulable under the Borda prize vector.*

Proof. Consider the tournament T that consists of teams A, B, C, and D, where A beats B and C, B beats C and D, C beats D and D beats A. We assert that B and D can collude. There are three possibilities to consider, each occurring w.p. $1/3$:

- A is paired with B, and C with D. Regardless of collusion, B and D both lose the first round, and their combined rank is $3+4=7$
- A is paired with C, and B with D. Under T, B goes on to lose against A and D goes on to lose against C, yielding combined rank $2+4=6$. If B throws so D wins, then B goes on to win against C and D goes on to win against A, yielding combined rank $1+3=4$.
- A is paired with D, and B with C. Regardless of collusion, B and D both win the first round, and their combined rank is $1+2=3$

Thus, if B and D play normally, we have it that their expected rank is $\frac{1}{3}(7+6+3) = 16/3$. If they collude, their expected rank is $\frac{1}{3}(7+4+3) = 14/3$. Thus, because total payoff under the Borda prize vector is proportional to expected rank [1], B and D strictly gain from this manipulation. □

2.2 Partial Results

As mentioned above, it has been proven that no tournament rule is 2-SNM-α for any $\alpha < \frac{1}{3}$ for the prize vector $\langle 1, 0, 0, \ldots, 0 \rangle$ and therefore no tournament rule is 2-SNM-α for any $\alpha < \frac{1}{3}$ for every possible prize vector [1]. Previous work found that NRKotH was optimal in that it is 2-SNM-$\frac{1}{3}$ for general prize vectors [1]. The natural question is therefore whether RCB is 2-SNM-$\frac{1}{3}$ for general prize vectors. We were unable to prove (or disprove) this claim, so we share our progress and some empirical evidence to show that this might be an interesting question for future work.

2.2.1 Proofs

In [3], the authors use a coupling argument to show that RSEB is 2-SNM-$\frac{1}{3}$ for tournaments with one winner, and we attempted to use this approach to examine RCB (which is based on RSEB). Our argument which loosely follows theirs is as follows:

Let T be a tournament over n teams, and suppose that arbitrary teams i, j consider whether to collude. Consider β, the set of all possible bracket initializations for RCB (note that $|\beta| = n!$ since each possible bracket corresponds to an ordering of the n teams). Then let B_{ij} be the set of all $B \in \beta$ such that i and j increase their collective prize winnings when RCB is initialized with bracket B.

Lemma 10. *For any $0 < \alpha < 1$, if $\frac{|B_{ij}|}{|\beta|} \leq \alpha$ for every i, j under RCB, then RCB is 2-SNM-α on general prize vectors.*

Proof. In a given bracket $B \in \beta$, if i and j collude in bracket B and increase their collective prize winnings when RCB is initialized to bracket B, by Lemma 1, we know that they increase their collective prize winnings by at most 1. Since an initialization for RCB is chosen uniformly at random, if $\frac{|B_{ij}|}{|\beta|} \leq \alpha$, then there is at most an α chance that i and j can increase their collective prize by colluding, meaning that the expected increase in their collective prize is at most α, implying that RCB is 2-SNM-α. □

Having proved Lemma 6, let $B' = \beta \setminus B_{ij}$ (that is, B' is the set of initial brackets where i and j do not increase their collective rewards by colluding). Our proof approach, following the one found in [3], is to construct injective maps $\sigma : B_{ij} \to B'$. Consider first an alternate proof for theorem 2:

Theorem 11. *RCB is 2-SNM-$\frac{1}{2}$ on general prize vectors (an alternative proof for theorem 2)*

Proof. Define the map $\sigma_1 : B_{ij} \to B'$ in the following way: for any bracket $B \in B_{ij}$, we know that, in B, i and j increase their collective prize winnings by colluding. As seen earlier, this is impossible if i and j are on different halves of the bracket (that is, if one of them is in position [1, n/2] and the other is in position [n/2 + 1, n]). Suppose that j is in position k_j Let B^* be the bracket where each team is in the same position as in bracket B except that j switched with the team at position $(k_j + n/2) \mod n$. We define $\sigma_1(B) = B^*$.

Note that σ_1 is injective because we can define its inverse; namely, for any B^* in the image of σ_1, we can recover a unique $B \in B_{ij}$ such that $\sigma_1(B) = B^*$ by swapping j in B^* with the element that is $\frac{n}{2}$ positions away from it. Since an injection $B_{ij} \to B^*$ exists, it follows that $|B_{ij}| \leq B^*$. Since $\beta = B_{ij} \cup B^*$, and $B_{ij} \cap B^* = \emptyset$, it follows that $\frac{|B_{ij}|}{|\beta|} \leq \frac{1}{2}$. By Lemma 6, this proves the theorem. □

One could imagine then a similar proof that RCB is 2-SNM-$\frac{1}{3}$. We present the beginnings of such a proof: though we were unable to complete the proof rigorously, it seems like a potentially promising approach that could perhaps prove fruitful in the future:

Conjecture 12. *RCB is 2-SNM-$\frac{1}{3}$ on general prize vectors.*

Proof. To prove the theorem, it suffices to find two injective maps $\sigma_1, \sigma_2 : B_{ij} \to B'$ with disjoint images since this would demonstrate that $\frac{|B_{ij}|}{|\beta|} \leq \frac{|B_{ij}|}{|B_{ij} \cup \text{Im}(\sigma_1) \cup \text{Im}(\sigma_2)|} \leq \frac{1}{3}$. First, define σ_1 to be the injective map described in Theorem 7. To prove the theorem, it suffices to find an injective map $\sigma_2 : B_{ij} \to B'$ such that the image of σ_2 does not share any elements with the image of σ_1. Unfortunately, we were unable to do this - nevertheless, here is an attempt that seems to come close (under some metric of closeness):

For some bracket $B \in B_{ij}$, define σ_2 using the following two cases:

Case 1: i and j play each other in the first round under bracket B.

In this case, let i be at position k_i and j be at position k_j in bracket B. Since $B \in B_{ij}$, we know that i and j are in the same half of bracket B. Let B' be the bracket that is identical to B except that j is swapped with the team at position $(k_i + \frac{n}{2}) \mod n$, and we define $\sigma_2(B) = B'$. Since i and j are on opposite sides of B', it follows that $B' \in B^*$

Note first that σ_2 is injective in this case since given any B' in the image of σ_2, we can find a unique $B \in B_{ij}$ such that $\sigma_2(B) = B'$ by swapping j with the team who plays i in the first round under B' (Note that, to complete the proof, the definition of σ_2 in the next case must not interfere with σ_2's injectivity in this case). Further, note that the image of B' in this case is disjoint from the image of σ_1 - in the image of σ_1, i and j are never at positions exactly $\frac{n}{2}$ away from each other, whereas in the image of σ_2 in this case, this is always true.

Case 2: i and j do not play each other in the first round under bracket B.

In this case, we were unable to define σ_2 such that it was both injective and had an image that was disjoint from the image of σ_2 in case 1 and the image of σ_1. By Lemma 7, we know that, since i and j increase their collective prize winnings in B since $B \in B_{ij}$, there must exist some team x such that a) one of $\{i, j\}$ wins against x and the other loses against x and b) x must play i or j in bracket B (such that they can increase their winnings by colluding so that one of them will win against x). This seems like an important fact that could be leveraged to prove the claim, but we were unable to do so within our project timeline. It is possible that part of the proof of Theorem 3.3 in [3] (particularly the authors' second mapping) could be adapted to solve this issue, but we could not figure out how to adapt that approach to the context of RCB. □

2.2.2 Empirical Evidence

Given that we could not find a sufficient proof for Theorem 12, we also endeavoured to find a counterexample (i.e. prove that RCB is not 2-SNM-$\frac{1}{3}$ for general prize vectors by finding a prize vector and tournament where two teams could collude and gain more than $\frac{1}{3}$ in expectation). We were unable to do so by finding special cases ourselves, so we created a simulation to potentially discover a counterexample: our code can be found at https://github.com/Ryan-S-M/COS521_FinalProject.

To attempt to discover a counter-example to Theorem 15, our simulation runs an exhaustive search over all tournaments on all possible brackets for all pairs of teams for a given prize vector or set of prize vectors to discover a tournament, bracket, and prize vector where two teams can collude to gain more than 1/3 in expectation. Note that this is a computationally expensive operation: there are $2^{\binom{n}{2}}$ possible tournaments (because the match result between each pair can be reversed) and $n!$ starting positions, and each must be evaluated for $\binom{n}{2}$ pairs. Even when taking advantages of symmetries in starting positions, this means that there are over 84 billion cases for 28 pairs of teams when $n = 8$ (and all this to only test 1 prize vector). For $n = 4$, our simulation ran quickly enough to examine 1000 random prize vectors for every possible tournament. We did not find a counterexample to the theorem for $n = 4$, though this simulation (run on the Borda vector) was how we discovered our proof that the Borda vector is manipulable under RCB and RRB. For $n = 8$, we ran our simulation on the Princeton research computing cluster for 24 hours without finding a counterexample, though the search space was too large to complete the simulation for as many prize vectors. Based on this empirical evidence, it seems somewhat likely that the theorem is provable, and perhaps future work can address this.

3 Fairness

Below we show that RCB and RRB are not cover-consistent or consistent under expectation. However, we note that they meet a basic standard for fairness in that they are both Condorcet-Consistent, meaning whenever an undefeated team exists, they are receive the best prize. Moreover, we note that bracket-based tournament rules are fair in that every team can play a match before anybody is required to play their next match again, which is not the case in many other rules such as Nested Randomized King of the Hill, where the pivot must play several teams in a row before the tournament can proceed.

Theorem 13. *RCB and RRB are not cover-consistent.*

Proof. Let r be either of RCB and RRB. Consider a tournament T^* with 4 teams (A, B, C, and D) with the following results: D beats everyone, A beats B and C, and B beats C. Note that A covers B, because it beats B and every team that B beats (i.e. C). Note also that there is a $1/3$ probability that the bracket pairs A with D and B with C. In this case, B beats C and loses to D to get rank 2, whereas A loses to D and then beats C to get rank 3. Thus, we have it that $P[\sigma^r_{T^*}(A) < \sigma^r_{T^*}(B)] \leq 2/3 < 1$; that is, even though A covers B, it is not ranked ahead of B with probability 1. Thus, r is not cover-consistent. □

Theorem 14. *RCB and RRB are not consistent under expectation.*

Proof. Let r be either of RCB and RRB. Consider a tournament T^* with 4 teams (A, B, C, and D) with the following results: A beats everyone except B, B loses to everyone except A, and D beats C. we have the following possibilities, each w.p. $1/3$:

- A plays B and C plays D, which results in A getting rank 3
- A plays C and B plays D, which results in A getting rank 1
- A plays D and B plays C, which results in A getting rank 1

We then have it that

$$\begin{aligned} \mathbb{E}[\sigma^r_{T^*}(A)] &= (1/3)\cdot 1 + (1/3)\cdot 1 + (1/3)\cdot 3 \\ &= \frac{5}{3} \\ &\neq 2 = n - |w_{T^*}(A)| \end{aligned}$$

□

4 Conclusions

In this paper, we extend Randomized Single Elimination Bracket to tournaments with multiple prizes, creating two new tournament ranking rules, *Randomized Complete Bracket (RCB)* and *Randomized Recursive Bracket (RRB)*. Unlike, NRKotH, RCB and RRB force all teams to play an equal number of games ($\log_2 n$), a potentially desirable property. We show that RCB is 2-SNM-$\frac{1}{2}$ and that RRB and RCN are both 2-SNM-$\frac{1}{3}$ for *Binary Power-of-Two Prize Vectors*. However, both RRB and RCB are manipulable under the Borda vector, and neither is cover-consistent or consistent under expectation (intuitively implying that these rules are not particularly “fair"). Finally, we provide a promising partial proof and empirical results towards showing that Randomized Complete Bracket is 2-SNM-$\frac{1}{3}$ for all prize vectors. Given the strength of our partial and empirical results, we hope future work can show that RCB and RRB are 2-SNM-$\frac{1}{3}$, making it an alternative to NRKotH with a fixed number of games for each team.

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